

SPECIAL LAGRANGIAN CONES

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ABSTRACT. We study homogeneous special Lagrangian cones in \mathbb{C}^n with isolated singularities. Our main result constructs an infinite family of special Lagrangian cones in \mathbb{C}^3 each of which has a toroidal link. We obtain a detailed geometric description of these tori. We prove a regularity result for special Lagrangian cones in \mathbb{C}^3 with a spherical link – any such cone must be a plane. We also construct a one-parameter family of asymptotically conical special Lagrangian submanifolds from any special Lagrangian cone.

1. INTRODUCTION

Let Y be Calabi-Yau manifold of complex dimension n with Kähler form ω and non-zero parallel holomorphic n -form Ω satisfying the normalization condition $\omega^n/n! = (-1)^{n(n-1)/2}(i/2)\Omega\wedge\bar{\Omega}$. Then $\operatorname{Re}(\Omega)$ is a calibrated form, whose calibrated submanifolds are called special Lagrangian submanifolds [6].

Moduli spaces of special Lagrangian submanifolds (and of other calibrated submanifolds) have appeared recently in string theory [1], [11]. On physical grounds, Strominger, Yau and Zaslow argued [11] that a Calabi-Yau manifold Y with a mirror partner \hat{Y} admits a (singular) fibration by special Lagrangian tori, and that \hat{Y} should be obtained by compactifying the dual fibration. To make this idea rigorous one needs to have control over the singularities and compactness properties of families of special Lagrangian submanifolds. In dimensions three and higher these properties are not well understood.

Motivated by these problems we study the simplest isolated singularities of special Lagrangian varieties – homogeneous cones in \mathbb{C}^n with an isolated singularity. These are also local models for more general singularities in that they are possible tangent cones to special Lagrangian currents at singular points.

Date: May 16, 2000.

1991 Mathematics Subject Classification. Primary 53C38; Secondary 53C43.

Key words and phrases. Differential geometry, isolated singularities, calibrated geometry, minimal submanifolds.

The author was supported in part by the States of Guernsey Education Council. He would like to thank the Institute for Advanced Study and Pomona College for their hospitality.

We introduce the notion of a θ -special Legendrian submanifold – a special class of minimal $(n - 1)$ -dimensional submanifolds – of $S^{2n-1}(1)$, and characterize θ -special Lagrangian cones in \mathbb{C}^n as those cones C whose links $L = C \cap S^{2n-1}$ are θ -special Legendrian submanifolds of S^{2n-1} (Proposition 2.5). From any special Legendrian link, in addition to a special Lagrangian cone, we obtain a one-parameter family of asymptotically conical special Lagrangian (possibly immersed) submanifolds.

Theorem A. *Let Σ^{n-1} be a θ -special Legendrian submanifold of $S^{2n-1}(1) \subset \mathbb{C}^n$. Let Σ_d ($d \in \mathbb{R}$) denote the set*

$$\left\{ (zp \in \mathbb{C}^n) : p \in \Sigma, z \in \mathbb{C}, \text{ where } \operatorname{Im}(z^n) = d, \arg z \in [0, \frac{\pi}{n}] \right\}.$$

Then

- (i) Σ_d is a θ -special Lagrangian variety.
- (ii) $\Sigma_0 = C(\Sigma) \cup C(e^{i\pi/n}\Sigma)$ where $C(\Sigma)$ denotes the cone on Σ .
- (iii) Σ_d is asymptotically conical, with two ends Σ and $e^{i\pi/n}\Sigma$.

In the case of special Lagrangian cones in \mathbb{C}^3 , results of Yau [15] and others put restrictions on three-dimensional special Lagrangian cones. For example, we obtain:

Theorem B. *Let C be a homogeneous special Lagrangian cone in \mathbb{C}^3 , with $L = C \cap S^5(1)$ a (possibly immersed) sphere. Then C must be a special Lagrangian plane.*

A simple corollary of this theorem is a regularity result for homogeneous solutions of the special Lagrangian graph equation in dimension three. The theorem is also sharp in the following two senses. The analogous statement in \mathbb{C}^4 is false, as recent examples of Chen *et al.* [2] demonstrate. Moreover, if the link type is a torus not a sphere then even in dimension three there are nontrivial special Lagrangian cones. Our main result gives an abundance of such cones.

Theorem C. *There exists a countably infinite family of non-isometric special Lagrangian cones in \mathbb{C}^3 . Each cone has link an embedded torus which is invariant under some $S^1 \subset SU(3)$.*

Each special Lagrangian cone in \mathbb{C}^3 also gives rise to other calibrated cones (Lemma 2.9). For example, from each special Lagrangian cone in \mathbb{C}^3 with an isolated singularity we associate: an associative cone in \mathbb{R}^7 with an isolated singularity, a coassociative cone in \mathbb{R}^7 and a Cayley cone in \mathbb{C}^4 with a line of singularities and special Lagrangian cones in \mathbb{C}^{n+3} with singularities along a real n -plane.

The strategy for the construction of the special Lagrangian cones in \mathbb{C}^3 is as follows. By exploiting the connection between harmonic maps and minimal surfaces we construct a two-parameter family $u_{\alpha,J}$ of special Legendrian immersions $\mathbb{R}^2 \rightarrow S^5(1)$. Harmonic maps from two dimensional domains to Lie groups and symmetric spaces have a rich structure, with relations to

infinite dimensional completely integrable systems and loop groups [5], [12]. In the case of S^1 -equivariant harmonic maps to spheres, a finite dimensional completely integrable system – the C. Neumann system – appears. Several geometric features of the harmonic map have nice interpretations in terms of conserved quantities of this system. The mechanical interpretation of the Legendrian condition is not so clear, but nonetheless we are able to obtain minimal Legendrian immersions from certain solutions of the Neumann system.

Theorem D. *For each $\theta \in [0, 2\pi)$ there exists a 2-parameter family $u_{\alpha,J}$, $(\alpha, J) \in [0, 1] \times [0, 1/3\sqrt{3}]$, of θ -special Legendrian immersions $\mathbb{R}^2 \rightarrow S^5(1)$ with the following properties:*

- (i) *The immersion $u_{\alpha,J}$ is invariant under the 1-parameter subgroup of $SU(3)$ generated by $A = \text{diag } i(1, \alpha, -1 - \alpha) \in su(3)$.*
- (ii) *For $\alpha = 1$ and $J = 1/3\sqrt{3}$ these immersions all describe Clifford tori, but otherwise all the immersions are geometrically distinct.*
- (iii) *The family $u_{\alpha,J}$ contains all θ -special Legendrian Legendrian immersions of the form (3.4) which cover a torus.*

To obtain tori from these immersions, in Proposition 5.3 we examine the conditions under which $u_{\alpha,J}$ is doubly periodic with respect to some lattice. By examining the special cases $u_{J,0}$ and $u_{0,\alpha}$ we deduce

Theorem E. (i) *For $\alpha \in \mathbb{Q} \cap (0, 1]$, the immersion $u_{0,\alpha}$ is doubly periodic and hence gives rise to a minimal Legendrian torus.*
(ii) *For a dense set of $J \in (0, 1/3\sqrt{3})$ the immersion $u_{J,0}$ is doubly periodic and hence gives rise to a minimal Legendrian torus.*

For the family $u_{0,\alpha}$, referred to in part (i) of the previous theorem, we give detailed information about the geometry (e.g. conformal structure, maximum and minimum values of the Gauss curvature and embeddedness) of the corresponding surfaces. As a corollary we find (Theorem 5.5) that there are embedded ‘almost flat’ minimal Legendrian tori. These tori demonstrate sharpness of two pinching results of Yau on minimal Lagrangian (Legendrian) immersions into $\mathbb{C}P^2(S^5)$.

The paper is organized as follows. In Section 2 we recall basic facts about special Lagrangian geometry in \mathbb{C}^n , introduce the notion of special Legendrian in S^{2n-1} and characterize special Lagrangian cones in terms of special Legendrian links. In Section 3 we recall basic facts from harmonic map theory: principally the relation with minimal surfaces and the appearance of the C. Neumann system in S^1 -equivariant harmonic maps into spheres. In Section 4 we study which solutions of the Neumann system give rise to special Legendrian immersions, give explicit parametrisations of these solutions and study the geometry of these immersions. In Section 5 we study the periodicity conditions for these immersions and hence are able to deduce our main results.

2. SPECIAL LAGRANGIAN CONES IN \mathbb{C}^n

2.1. Special Lagrangian geometry in \mathbb{C}^n . Special Lagrangian geometry is an example of a *calibrated geometry* [6]. We review some elementary facts about calibrations and special Lagrangian geometries in \mathbb{C}^n in particular (see [6] for further details).

Each calibrated geometry is a distinguished class of minimal submanifolds of a Riemannian manifold (M, g) associated with a closed differential p -form ϕ of comass one.

For each $m \in M$, the *comass* of ϕ is defined to be

$$\|\phi\|_m^* = \sup\{\langle \phi_m, \xi_m \rangle : \xi_m \text{ is a unit simple } p\text{-vector at } m\}.$$

In other words, $\|\phi\|_m^*$ is the supremum of ϕ restricted to the Grassman of oriented p -dimensional planes $G(p, T_m M)$, regarded as a subset of $\Lambda^p T_m M$.

To any form of comass one there is a natural subset of $G(p, TM)$

$$G_m(\phi) = \{\xi_m \in G(p, T_m M) : \langle \phi_m, \xi_m \rangle = 1\},$$

that is, the collection of oriented p -planes on which ϕ assumes its maximum. These planes are the planes *calibrated by ϕ* . An oriented p -dimensional submanifold of (M, g) is *calibrated by ϕ* if its tangent plane at each point is calibrated.

The key property of calibrated submanifolds is that they are *homologically volume minimizing*

Lemma 2.1 (Harvey and Lawson [6]). *Let (M, g, ϕ) be a calibrated geometry, and suppose S is a calibrated submanifold (possibly with boundary). Then for any oriented p -dimensional submanifold \hat{S} homologous to S*

$$\text{vol}(S) \leq \text{vol}(\hat{S})$$

with equality if and only if \hat{S} is also calibrated (by ϕ).

Let z_1, \dots, z_n denote standard complex coordinates on \mathbb{C}^n . For any $\theta \in [0, 2\pi)$ the real n -form

$$\alpha_\theta = \text{Re}(e^{i\theta} dz^1 \wedge \dots \wedge dz^n)$$

is a calibrated form, called the θ -special Lagrangian calibration on \mathbb{C}^n .

For the proof that α_θ has comass one see [6]. A θ -special Lagrangian plane (we will sometimes abbreviate this as θ -SLG) is an oriented n -plane calibrated by the form α_θ . A useful characterization of the θ -special Lagrangian planes is

Lemma 2.2. *An oriented n -plane ξ in \mathbb{C}^n is θ -special Lagrangian (for the correct choice of orientation) if and only if*

1. ξ is Lagrangian with respect to the standard symplectic form $\omega = \sum dx^i \wedge dy^i$, (i.e. ω restricts to zero on ξ) and
2. $\beta_\theta := \text{Im}(e^{i\theta} dz^1 \wedge \dots \wedge dz^n)$ restricts to zero on ξ .

One reason for considering the whole S^1 -family of special Lagrangian calibrations is the following result of Harvey and Lawson (Proposition 2.17 of [6]):

Proposition 2.3. *A connected oriented Lagrangian submanifold $S \subset \mathbb{C}^n$ is minimal (i.e. it is a critical point of volume, or its mean curvature H vanishes) if and only if S is θ -special Lagrangian for some θ .*

2.2. Regular cones and special Legendrian links. For any compact connected oriented embedded submanifold $\Sigma \subset S^{n-1}(1) \subset \mathbb{R}^n$ define the cone on Σ ,

$$C(\Sigma) = \{tx : t \in \mathbb{R}^{\geq 0}, x \in \Sigma\}.$$

A cone C in \mathbb{R}^n is *regular* if there exists Σ as above so that $C = C(\Sigma)$, in which case we call Σ the *link* of the cone C . $C(\Sigma) - 0$ is an embedded smooth submanifold, but $C(\Sigma)$ has an isolated singularity at 0 unless Σ is a totally geodesic sphere.

To characterize the links of regular special Lagrangian cones we need to introduce some geometric structures on the unit sphere S^{2n-1} in \mathbb{C}^n . As a convex hypersurface in a Kähler manifold [8], $S^{2n-1}(1)$ inherits a *contact form*, that is, a 1-form γ so that

$$(2.1) \quad \gamma \wedge d\gamma^{n-1} \neq 0.$$

Let X denote the Euler vector field $x \cdot \partial / \partial x$ on \mathbb{C}^n and ω denote the standard symplectic form on \mathbb{C}^n . Then the contact form on $S^{2n-1}(1)$ is

$$\gamma = \iota_X \omega|_{S^{2n-1}}.$$

Associated with γ is the *contact distribution*, the hyperplane field $\ker \gamma \subset TS^{2n-1}$. The condition (2.1) on γ ensures that the distribution $\ker \gamma$ is not integrable. The maximal dimensional integral submanifolds (i.e. submanifolds on which γ restricts to zero) of the distribution are $(n-1)$ -dimensional and are called *Legendrian submanifolds*.

The relevance of Legendrian submanifolds of the sphere can be seen from the next result whose proof is standard.

Lemma 2.4. *Let Σ be an $(n-1)$ -dimensional submanifold of $S^{2n-1}(1)$. Then $C(\Sigma)$ is Lagrangian if and only if Σ is Legendrian.*

For any p -form ϕ on \mathbb{R}^n define the *normal part* of ϕ by

$$\phi_N = \iota_X \phi,$$

where X again denotes the Euler vector field on \mathbb{C}^n . In particular, $\alpha_{\theta,N}$ denotes the normal part of the θ -special Lagrangian calibration α_θ .

An oriented $(n-1)$ -dimensional submanifold Σ of $S^{2n-1}(1)$ is a *θ -special Legendrian submanifold* if at each point of Σ , $\alpha_{\theta,N}$ restricts to the volume form on Σ .

Proposition 2.5. *A regular cone $C = C(\Sigma)$ in \mathbb{C}^n is θ -special Lagrangian if and only if Σ is θ -special Legendrian.*

Proof. This is essentially a special case of Theorem 5.6 of [6]. We sketch the proof. For any constant p -form ϕ on \mathbb{R}^n define the *tangential part* of ϕ to be

$$\phi_T = \iota_X \left(\frac{x}{|x|} \cdot dx \wedge \phi \right).$$

Then ϕ decomposes as

$$(2.2) \quad \phi = \phi_T + \frac{x}{|x|} \cdot dx \wedge \phi_N$$

where ϕ_N is the normal part of ϕ defined previously. When $d\phi = 0$, restricting to the unit sphere it follows that

$$d\phi_T = 0 \quad \text{and} \quad d\phi_N = p\phi_T.$$

Since $\phi_N(\xi) = \phi(x \wedge \xi)$ and $\|\xi\| = \|x \wedge \xi\|$ for any simple $(p-1)$ -vector in $\Lambda^{p-1}x^\perp$, ϕ_N still has comass one (but since ϕ_N is not closed it is not a calibration itself). Moreover, submanifolds Σ of $S^{n-1}(1)$ on which ϕ_N restricts to the volume form are exactly those for which $C(\Sigma)$ is calibrated by ϕ . Hence the result follows by taking ϕ to be any of the θ -special Lagrangian calibrations α_θ . \square

We also have the following Legendrian analogue of Proposition 2.3

Proposition 2.6. *A connected oriented Legendrian submanifold of $S^{2n-1}(1)$ is minimal if and only if it is θ -special Legendrian for some θ .*

Proof. Let Σ be a minimal Legendrian submanifold of $S^{2n-1}(1)$. It is a standard fact [10] that $C(\Sigma)$ is minimal if and only if Σ is minimal in the unit sphere. Thus $C(\Sigma)$ is a minimal Lagrangian cone which from Proposition 2.3 must be θ -special Lagrangian for some θ . By the previous proposition this implies Σ is θ -special Legendrian. The converse is similar. \square

We finish the section by proving Theorem A, which gives a one-parameter family of asymptotically conical special Lagrangian varieties modeled on any special Lagrangian cone. This result generalizes Theorem 3.5 of [6] which is our result in the special case $\Sigma = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_i \in \mathbb{R} \text{ with } \sum x_i^2 = 1\}$.

Proof of Theorem A. (i) By Proposition 2.5, $C(\Sigma)$ is a θ -SLG cone. By rotating Σ by $A = \text{diag}(e^{-i\theta/n}, \dots, e^{-i\theta/n})$ we can assume $C(\Sigma)$ is 0-SLG. Thus $\beta|_{C(\Sigma)} = 0$. Let $\phi : \Sigma \rightarrow S^{2n-1}$ denote the inclusion of Σ in the sphere, and let x_1, \dots, x_{n-1} be local coordinates on Σ . Then $\beta|_{C(\Sigma)} = 0$ is equivalent to

$$(2.3) \quad \text{Im} \left(\det_{\mathbb{C}} \left(\phi, \frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_{n-1}} \right) \right) = 0.$$

Let $\Phi : \mathbb{R} \times \Sigma \rightarrow \mathbb{C}^n$ be given by $\Phi(t, x) = f(t)\phi(x)$ where $f : \mathbb{R} \rightarrow \mathbb{C}$ is some nonconstant smooth complex valued function. It is straightforward to check that any such Φ gives rise to a Lagrangian immersion to \mathbb{C}^n .

Let $x_0 = t$ and for $i = 0, \dots, n-1$ denote $\partial\Phi/\partial x_i$ by Φ_i . Then for $j = 1, \dots, n-1$

$$\omega(\Phi_0, \Phi_j) = \omega(\dot{f}\phi, f\phi_j) = \operatorname{Re}(\bar{f}\dot{f}) \omega(\phi, \phi_i) + \operatorname{Im}(\bar{f}\dot{f}) \langle \phi, \phi_i \rangle = 0$$

where the first term vanishes because ϕ is Legendrian and the second because $|\phi|^2 = 1$. For $j, k = 1, \dots, n-1$ we have

$$\omega(\Phi_j, \Phi_k) = \omega(f\phi_j, f\phi_k) = |f|^2 \omega(\phi_j, \phi_k) = 0$$

and so Φ is Lagrangian as claimed.

Now we claim that Φ is 0-SLG if and only if f satisfies

$$(2.4) \quad \operatorname{Im}(f^n) = d$$

for some real constant d . To prove this it is enough to show that $\beta|_\Phi = 0$ holds if and only if (2.4) is satisfied. But $\beta|_\Phi = 0$ is equivalent to

$$(2.5) \quad \operatorname{Im}(\det_{\mathbb{C}}(\Phi_0, \dots, \Phi_{n-1})) = 0.$$

Since $C(\Sigma)$ is 0-SLG, we have $\operatorname{Im}(\det_{\mathbb{C}}(\phi, \phi_1, \dots, \phi_{n-1})) = 0$ and hence

$$\begin{aligned} \operatorname{Im}(\det_{\mathbb{C}}(\Phi_0, \dots, \Phi_{n-1})) &= \operatorname{Im}\left(\det_{\mathbb{C}}(\dot{f}\phi, f\phi_1, \dots, f\phi_{n-1})\right) \\ &= \operatorname{Im}\left(f f^{n-1} \det_{\mathbb{C}}(\phi, \phi_1, \dots, \phi_{n-1})\right) \\ &= \operatorname{Re}(\det_{\mathbb{C}}(\phi, \phi_1, \dots, \phi_{n-1})) \times \operatorname{Im}(f f^{n-1}). \end{aligned}$$

Thus (2.5) holds if and only if

$$\operatorname{Im}(f f^{n-1}) = \frac{1}{n} \left(\operatorname{Im} \frac{d}{dt} f^n \right) = \frac{1}{n} \frac{d}{dt} \operatorname{Im}(f^n) = 0.$$

Hence Φ is 0-SLG if and only if $\operatorname{Im}(f^n) = d$ as claimed.

Parts (ii) and (iii) are straightforward to verify. \square

2.3. Minimal Legendrian surfaces. In dimension two, any special Lagrangian cone must be a union of special Lagrangian planes (since its link must be a union of Legendrian geodesics in S^3). In the first interesting case, namely special Lagrangian cones in \mathbb{C}^3 , restrictions on the geometry and topology of the allowable links follow from the next result, essentially due to Yau.

Theorem 2.7. [14, 15] *Let Σ be a minimal Legendrian surface of $S^5(1)$. Then:*

- (i) *If Σ has genus zero, Σ is totally geodesic.*
- (ii) *If Σ is a complete nonnegatively curved surface, Σ is a totally geodesic sphere or a flat torus.*
- (iii) *If Σ is complete nonpositively curved surface then Σ is a flat torus.*

This theorem is the Legendrian analogue of a result of Yau on minimal Lagrangian immersions into Kähler surfaces of constant holomorphic sectional curvature (e.g. $\mathbb{C}P^2$ with the Fubini-Study metric). In fact, using

Reckziegel's observation [9] about the local correspondence between minimal Legendrian immersions into S^5 and minimal Lagrangian immersions into $\mathbb{C}P^2$, one can deduce the Legendrian result from the Lagrangian one.

As a corollary of part (i) of Theorem 2.7 we deduce Theorem B. Applying this theorem to the special case of special Lagrangian graphs we deduce

Corollary 2.8. *Any homogeneous degree 1 solution u of the 3-dimensional special Lagrangian graph equation*

$$\Delta u = \det \text{Hess}(u)$$

is a quadratic function.

Theorem B is sharp in the following two senses. Firstly, in \mathbb{C}^4 the analogous result is false as recent examples of Chen *et al.* show [2]. Secondly, in \mathbb{C}^3 there are nontrivial special Lagrangian cones with link type a torus, the simplest example of which is the cone on a generalized Clifford torus.

Let T be the Lagrangian product 3-torus contained in $S^5(1)$

$$T = \{z \in \mathbb{C}^3 : |z_i|^2 = 1/3, \quad i = 1, 2, 3\},$$

and T_θ be the 2-torus

$$T_\theta = \{z \in T : \sum \arg z_i = \theta\}.$$

The T_θ , the *generalized Clifford tori*, are all flat minimal Legendrian tori and $T_{\theta/3}$, $T_{\pi+\theta/3}$ are θ -special Legendrian. Harvey and Lawson discovered the cones on these tori in a family of special Lagrangian level sets invariant under the maximal torus $T^2 \subset \text{SU}(3)$. This high degree of symmetry allowed them to explicitly write down solutions. In the next three sections we shall find a family of nonisometric minimal Legendrian tori in $S^5(1)$, which are invariant under an $S^1 \subset \text{SU}(3)$. This symmetry is still enough to allow us to give quite explicit descriptions of these tori.

Special Lagrangian cones in dimension three with isolated singularities, also give rise naturally to several related singular calibrated varieties. For example, remarks of Harvey-Lawson [6] (IV.2.C. Remark 2.12) and Donaldson-Thomas [3] show the following:

Lemma 2.9. *If X^3 is a 0-special Lagrangian variety, then:*

- (i) $X \times \{pt\} \subset \mathbb{C}^3 \times \mathbb{R}$ *is an associative variety*
- (ii) $X \times \mathbb{R} \subset \mathbb{C}^3 \times \mathbb{R}$ *is a coassociative variety*
- (iii) $X \times \mathbb{R} \subset \mathbb{C}^3 \times \mathbb{C}$ *is a Cayley variety.*

In cases (ii) and (iii) starting with a special Lagrangian cone with an isolated singularity we obtain *cylindrical* cones, which have a whole line of singularities. One also gets special Lagrangian cylindrical cones in \mathbb{C}^{n+3} by taking the Cartesian product of a 3-dimensional cone in \mathbb{C}^3 with a real n -plane in \mathbb{C}^n .

3. HARMONIC MAPS, MINIMAL SURFACES AND THE NEUMANN SYSTEM

We shall construct S^1 -invariant minimal Legendrian tori in $S^5(1)$ by exploiting two relationships. The first is the connection between harmonic maps and minimal surfaces. The second is the link between S^1 -equivariant harmonic maps into spheres and the C. Neumann system describing motion on the sphere under a quadratic potential.

3.1. Harmonic Maps. We recall some definitions and basic facts from harmonic map theory. Suppose M and N are Riemannian manifolds. For any C^1 map $u : M \rightarrow N$ define a smooth function $e(u)$, the *energy density* of u , by $e(u)(x) = \text{Tr}(du_x^2)$. Define a functional on $C^1(M, N)$, the *total energy*, by $E(u) = \int_M e(u) \mu_M$, where μ_M is the Riemannian volume element of M . Critical points of E are *harmonic maps* from M into N .

If N is isometrically embedded in \mathbb{R}^K then we can view a function $u : M \rightarrow N$ as a function $u = (u^1, \dots, u^K)$ into \mathbb{R}^K with the constraint that $u(x) \in N$ for all $x \in M$. Then

$$(3.1) \quad E(u) = \sum_{i=1}^K \int_M |\nabla u^i|^2 \mu_M.$$

Extremals of E subject to the constraint that $u(M) \subset N$ give us the harmonic maps to N . From this we see that the harmonic map equations can be written simply as

$$\Delta u(x) \perp T_{u(x)}N, \quad u(x) \in N, \quad \forall x \in M.$$

In the case that $N = S^n(1) \subseteq \mathbb{R}^{n+1}$ (with the metric induced by this inclusion) this implies $\Delta u = \lambda u$ for some function λ on M . Taking the inner product of both sides with u and using the constraint equation $|u|^2 = 1$ we determine that $\lambda = (u, \Delta u) = -|du|^2$. Summarizing we have

Lemma 3.1. *A smooth map $u : M \rightarrow S^n(1) \subset \mathbb{R}^{n+1}$ is harmonic iff and only if u satisfies the equation*

$$(3.2) \quad \Delta u = -|du|^2 u.$$

Finally we recall what happens to the harmonic map equations when we make a conformal change of metric on the domain M . If $\tilde{g} = \lambda^2 g$ then $\tilde{g}^{-1} = \lambda^{-2} g^{-1}$, and $\tilde{\mu}_M = \lambda^m \mu_M$. Hence $E_{\tilde{g}}(u) = \lambda^{m-2} E_g(u)$ and we see that E is conformally invariant if and only if $\dim M = 2$. Therefore in dimension 2 harmonicity depends only on the structure of M as a Riemann surface. In particular there is a natural quadratic differential Φ , the *Hopf differential*. If z is a local complex coordinate on M then $\Phi = \phi(z) dz^2$ where

$$(3.3) \quad \phi(z) = (u_z, u_z) = \frac{1}{4} (|u_x|^2 - |u_y|^2 - 2i(u_x, u_y)).$$

Harmonicity of u implies that Φ is holomorphic. If the Hopf differential vanishes the map u is *conformal*. Moreover, we have the following connection with minimal surfaces:

Proposition 3.2. ([4]) *u is harmonic and conformal if and only if u is a (branched) minimal immersion.*

3.2. Equivariant Harmonic Maps and the Neumann System. For harmonic maps from \mathbb{R}^2 to $S^5(1) \subseteq \mathbb{C}^3$ (where both \mathbb{R}^2 and $S^5(1)$ are given their standard metrics) of the special form

$$(3.4) \quad u(s, t) = e^{As} z(t)$$

where $A \in so(6)$ and $z : \mathbb{R} \rightarrow S^5(1)$, it follows from (3.2) that u is harmonic if and only if z satisfies

$$(3.5) \quad \ddot{z} + A^2 z = -(|\dot{z}|^2 + |Az|^2)z$$

where $\dot{}$ denotes differentiation with respect to t . As Uhlenbeck noted [13] these are the equations of motion for the C. Neumann problem of motion of a particle on a sphere under the quadratic potential $|Az|^2$.

Define \mathbb{R} actions on \mathbb{R}^2 and $S^5(1)$ by

$$\begin{aligned} \gamma \cdot (s, t) &= (s + \gamma, t) \\ \gamma \cdot p &= e^{A\gamma} p \end{aligned}$$

where $s, t, \gamma \in \mathbb{R}$ and $p \in S^5(1)$. These induce an action in the usual manner on the Banach manifold $C^1(\mathbb{R}^2, S^5)$ by

$$(3.6) \quad (\gamma \cdot u)(x) = \gamma \cdot u(\gamma^{-1} \cdot x)$$

the fixed points of which are precisely maps of the form (3.4). Since \mathbb{R} acts by isometries on both \mathbb{R}^2 and S^5 , it follows from the definition of E that it is an \mathbb{R} -invariant function on $C^1(\mathbb{R}^2, S^5)$. Hence we could also appeal to Palais's Principle of Symmetric Criticality to find the equations satisfied by z , as in [13].

From now on we consider only the case that $A \in u(3)$, so that the one-parameter group e^{As} preserves both the metric and the symplectic structure. Then by conjugation we may assume that $A = \text{diag } i(\lambda_1, \lambda_2, \lambda_3)$. In this case equation (3.5) becomes

$$(3.7) \quad \ddot{z}_j - \lambda_j^2 z_j = -\lambda z_j, \quad z_j \in \mathbb{C}, \quad j = 1, 2, 3$$

where

$$(3.8) \quad \lambda = |Az|^2 + |\dot{z}|^2.$$

It will be convenient to rewrite the equations slightly. Writing $z_j = R_j e^{i\theta_j}$ we see that (3.7) is equivalent to

$$(3.9) \quad \ddot{R}_j - \frac{J_j^2}{R_j^3} = (\lambda_j^2 - \lambda) R_j, \quad j = 1, 2, 3$$

where θ_j is determined up to a constant by the relation $J_j = R_j^2 \dot{\theta}_j$.

There are some obvious conserved quantities. From conservation of energy we have

$$(3.10) \quad H = |\dot{z}|^2 - |Az|^2$$

and conservation of the quantities

$$(3.11) \quad J_j = x_j \dot{y}_j - y_j \dot{x}_j, \quad j = 1, 2, 3$$

expresses the fact that angular momentum in each of the three complex planes z_1, z_2, z_3 is conserved. For details of other less obvious conserved quantities of the Neumann system we refer the reader to [13].

The condition that u be conformal is conveniently expressed in terms of the integrals of motion. Namely, u is conformal if and only if

$$(3.12) \quad |u_s|^2 - |u_t|^2 = |\dot{z}|^2 - |Az|^2 = H = 0$$

and

$$(3.13) \quad (u_s, u_t) = (\dot{z}, Az) = \sum_{i=1}^3 \lambda_i J_i = 0.$$

Summarizing we have

Proposition 3.3. [13] *$u(s, t) = e^{As} z(t) : \mathbb{R}^2 \rightarrow S^5(1)$ is a minimal immersion if and only if z satisfies the equations of motion of the Neumann system (3.7) and the conserved quantities H, J_j satisfy the constraints (3.12) and (3.13).*

4. S^1 EQUIVARIANT MINIMAL LEGENDRIAN IMMERSIONS

4.1. The Legendrian constraints. For $u(s, t) = e^{As} z(t)$ to be a minimal Legendrian immersion, besides the conditions of Proposition 3.3, two further constraints must hold

$$(4.1) \quad \alpha(u_s) = \omega(u, u_s) = \omega(z, Az) = \sum_{i=1}^3 \lambda_i R_i^2 = 0,$$

and

$$(4.2) \quad \alpha(u_t) = \omega(u, u_t) = \omega(z, \dot{z}) = \sum_{i=1}^3 J_i = 0.$$

Note that the second equation corresponds merely to further constraints on the values of the integrals of the Neumann system. The first equation is more mysterious and is in general not preserved under the flow of the Neumann system. In fact, we have

Lemma 4.1. *There are minimal Legendrian immersions of the form given in (3.4) if and only if $A \in su(3)$.*

Proof. From Proposition 2.6 any minimal Legendrian immersion is θ -special Legendrian for some θ and hence β_θ restricts to zero on the cone. At a point xu on the cone (where $x \in \mathbb{R}^+$) we have

$$\beta_\theta|_{C(u)} = x^2 \text{Im} \left(e^{i\theta} \det_{\mathbb{C}}(u, u_s, u_t) \right) = x^2 \text{Im} \left(e^{i \sum \lambda_i s} e^{i\theta} \det_{\mathbb{C}}(z(t), Az(t), \dot{z}(t)) \right).$$

Since this must hold for all real s and t , for β_θ to restrict to zero we must have $\sum \lambda_i = 0$ as claimed.

One can also show necessity directly from the equations for a minimal Legendrian equation by showing that the constraints (3.12, 3.13, 4.1, 4.2) are not consistent with the equations of motion of the Neumann system unless $A \in su(3)$.

To see this let us compute the second derivative of the mysterious constraint $c := \omega(z, Az)$ for a solution of the Neumann system at an instant when all the constraints and their first derivatives are satisfied. One finds

$$\ddot{c} = \omega(Az, \ddot{z}) + \omega(A\dot{z}, \dot{z}) = -\omega(Az, A^2 z) + \omega(A\dot{z}, \dot{z}).$$

Let $c_1 = \omega(A^2 z, Az)$ and $c_2 = \omega(A\dot{z}, \dot{z})$. Then c_1 may be expressed in terms of the symmetric polynomials in the λ_i as

$$c_1 = \sum \lambda_i^3 R_i^2 = \left(\sum \lambda_j \right) \left(\sum \lambda_i^2 R_i^2 \right) - \left(\sum \lambda_j \lambda_k \right) \left(\sum \lambda_i R_i^2 \right) + \lambda_1 \lambda_2 \lambda_3 \left(\sum R_i^2 \right).$$

Hence using the constraints we have

$$c_1 = \left(\sum \lambda_j \right) |Az|^2 + \lambda_1 \lambda_2 \lambda_3.$$

A calculation shows

$$c_2 = -\lambda_1 \lambda_2 \lambda_3 \frac{|\dot{z}|^2}{|Az|^2}$$

and so

$$(4.3) \quad \ddot{c} = \left(\sum \lambda_j \right) |Az|^2 - H \frac{\lambda_1 \lambda_2 \lambda_3}{|Az|^2}.$$

Clearly once we have imposed the constraint $H = 0$, $\ddot{c} = 0$ if and only if $A \in su(3)$. Moreover, by differentiating (4.3) it is easy to verify that all higher derivatives of the constraint c also vanish when $A \in su(3)$. Thus to show existence of minimal Legendrian immersions we need only show there exist initial conditions for the Neumann system which satisfy all the constraints together with the first derivative of the mysterious constraint. We will see that this is indeed the case in the proof of Theorem D which we now give. \square

Proof of Theorem D. Let u be a minimal Legendrian immersion of the form (3.4), i.e. $u(s, t) = e^{As} z(t)$ where $A \in su(3)$. By conjugation we may assume $A = \text{iddiag}(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_1 \geq \lambda_2 \geq 0 > \lambda_3$. Let $\alpha = \lambda_2/\lambda_1$, then $\alpha \in [0, 1]$ and $A = \text{iddiag}(\lambda_1(1, \alpha, -1 - \alpha))$. Moreover, by rescaling s and t we may assume that $\lambda_1 = 1$. Let

$$\vec{I} = (1, 1, 1), \quad \vec{J} = (J_1, J_2, J_3), \quad \vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3), \quad \vec{R}^2 = (R_1^2, R_2^2, R_3^2).$$

Then the constraints (3.13,4.1,4.2) together with the constraint that z lie on the unit sphere can be written as

$$(4.4) \quad \vec{1} \cdot \vec{J} = 0, \quad \vec{\lambda} \cdot \vec{J} = 0,$$

$$(4.5) \quad \vec{1} \cdot \vec{R}^2 = 0, \quad \vec{\lambda} \cdot \vec{R}^2 = 1$$

and $A \in su(3)$ is equivalent to $\vec{1} \cdot \vec{\lambda} = 0$. Let $\vec{\mu}$ be the cross product of $\vec{1}$ and $\vec{\lambda}$

$$\vec{\mu} = \vec{1} \times \vec{\lambda} = (-1 - 2\alpha, 2 + \alpha, \alpha - 1).$$

The constraints in (4.4) are equivalent to

$$(4.6) \quad \vec{J} = J\vec{\mu}$$

for some constant J , while the constraints in (4.5) are equivalent to

$$(4.7) \quad \vec{R}^2(t) = \gamma(t)\vec{\mu} + \frac{1}{3}\vec{1}$$

for some function $\gamma(t)$. The remaining constraint $|\dot{z}|^2 = |Az|^2$ then becomes

$$(4.8) \quad \frac{\dot{\gamma}^2}{4} + J^2 = R_1^2 R_2^2 R_3^2$$

or in terms of γ

$$(4.9) \quad \frac{\dot{\gamma}^2}{4} + J^2 = \gamma^3 \mu_1 \mu_2 \mu_3 + \frac{\gamma^2}{3} \sum_{i \neq j} \mu_i \mu_j + \frac{1}{27}.$$

Since we seek periodic solutions we may assume that $\gamma(0) = \gamma_0 > 0$, $\dot{\gamma}(0) = 0$. Then at $t = 0$, (4.9) becomes

$$(4.10) \quad \gamma^3 \mu_1 \mu_2 \mu_3 + \frac{\gamma^2}{3} \sum_{i \neq j} \mu_i \mu_j + \frac{1}{27} = J^2$$

and thus specifying γ_0 determines J^2 (and vice versa).

Let us fix $\alpha \in [0, 1]$ and consider the case where $J \neq 0$. Given $J \in (0, \frac{1}{3\sqrt{3}}]$, (4.10) has a unique smallest nonnegative root $\gamma_+(J)$. Let $\gamma(0) = \gamma_+(J)$, $\dot{\gamma}(0) = 0$. Then up to a translation in time any periodic solution of (4.10) (except possibly a solution corresponding to $J = 0$ which we shall treat later) arises from such an initial condition. Once the initial conditions for γ and $\dot{\gamma}$ have been specified, (4.7) fixes $R_j(0)$ and $\dot{R}_j(0)$ for $j = 1, 2, 3$. Given J and α , (4.6) fixes \vec{J} . If we define $\dot{\theta}_j = J_j/R_j^2$, then $\dot{\theta}_j(0)$ is determined by (4.6) and (4.7) for $j = 1, 2, 3$. By a global rotation in $SU(3)$ (e.g. replacing $z(t)$ by $Bz(t)$ where $B = \exp(i \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)) \in SU(3)$) we may rotate $z(t)$ so that $\theta_2(0) = \theta_3(0) = 0$. We may not assume also that $\theta_1(0) = 0$ without allowing $B \in U(3)$ in which case we will change the value of θ for which u is θ -special Legendrian. In the case $J \neq 0$ we shall verify later that choosing $-\theta_1(0) = \theta$ gives rise to a θ -special Legendrian immersion (in the case $J = 0$ choosing $-\theta_1(0) = \theta + \pi/2$ gives rise to a θ -special Legendrian immersion).

Thus for each $\theta \in [0, 2\pi)$, and $(\alpha, J) \in [0, 1] \times (0, 1/3\sqrt{3}]$ there is a unique solution of the Neumann equation (given by specifying initial data in the manner above) which satisfies the constraints (3.12, 3.13, 4.1, 4.2). Hence by the proof of the previous lemma it gives rise to a minimal Legendrian immersion which we denote $u_{\alpha, J}$.

In the case $J = 0$ we will explicitly exhibit solutions later in this section and see that as $\alpha \rightarrow 0$ the period of γ becomes infinite, and that the limiting solution $u_{0,0}$ describes a minimal Legendrian sphere (which as previously noted is necessarily totally geodesic).

To see which immersions $u_{\alpha, J}$ are geometrically distinct consider in greater detail the geometry of these immersions. Since the immersions are all conformal, the metric g induced on \mathbb{R}^2 can be described by a single positive function $y = |Az|^2 = |\dot{z}|^2$, where $g = y|dz|^2$. A calculation shows that γ and y are related by

$$(4.11) \quad y = -\gamma\mu_1\mu_2\mu_3 + \frac{1}{3} \sum \lambda_i^2.$$

It follows from (4.11) and (4.9) that y satisfies

$$(4.12) \quad \dot{y}^2 + 4y^3 - 2y^2 \sum \lambda_i^2 = 4C$$

where

$$-C = \lambda_1^2 \lambda_2^2 \lambda_3^2 + J^2 \mu_1^2 \mu_2^2 \mu_3^2.$$

The Gauss curvature of the immersion satisfies

$$(4.13) \quad K = -\frac{(\ln y)''}{2y} = 1 + 2Cy^{-3}.$$

In the case $J = 1/3\sqrt{3}$, the corresponding solution of (4.9) is $\gamma \equiv 0$ independent of the choice of $\alpha \in [0, 1]$ and hence $u_{\alpha, 1/3\sqrt{3}}$ has $K \equiv 0$. It follows that u must be (a piece of) a generalized Clifford torus. Similarly, if $\alpha = 1$ then $\mu_3 = 0$ and it follows from (4.11) that $y \equiv 2$. Once again $K \equiv 0$ and hence $u_{1, J}$ is a (piece of a) generalized Clifford torus.

All other immersions $u_{\alpha, J}$ are geometrically distinct. To begin with, note that the remaining immersions are all invariant under a unique 1-parameter family of $SU(3)$ – the subgroup generated by $A = i \operatorname{diag}(1, \alpha, -1 - \alpha)$. For $\alpha \in [0, 1)$ these are all inequivalent, hence $u_{\alpha, J}$ and $u_{\tilde{\alpha}, \tilde{J}}$ are distinct when $\alpha \neq \tilde{\alpha}$. Now fix α and consider $u_{\alpha, J}$ for $J \in (0, 1/3\sqrt{3})$. We claim that the minimum and maximum values of the Gauss curvature K are respectively strictly decreasing and increasing functions of J on $(0, 1/3\sqrt{3})$. It follows that $u_{\alpha, J}$ and $u_{\alpha, \tilde{J}}$ are geometrically distinct when $\alpha \neq \tilde{\alpha}$.

To proof the previous claim, note that (4.13) shows that for a given immersion $u_{\alpha, J}$ the minimum (maximum) value of K occurs at the minimum (maximum) value of y . From (4.12) it is clear that for fixed α , y_{\min} and y_{\max} , the minimum and maximum values attained by y , are strictly decreasing and increasing functions of J respectively. Since C is a decreasing function of J , from (4.13) we see that the minimum and maximum values of

K are, like y , strictly decreasing and increasing functions of J respectively as claimed. \square

It is also possible to write down explicit solutions in terms of elliptic functions. Let us express γ in terms of the Jacobi elliptic functions. Recall that γ satisfies the equation

$$\frac{\dot{\gamma}^2}{4} + J^2 = \gamma^3 \mu_1 \mu_2 \mu_3 + \frac{\gamma^2}{3} \sum_{i \neq j} \mu_i \mu_j + \frac{1}{27}$$

and that for $J^2 \in [0, 1/27)$ and $\alpha \neq 1$ there are three solutions $\Gamma_1, \Gamma_2, \Gamma_3$ to this equation when $\dot{\gamma} = 0$. Let us label these solutions so that $\Gamma_2 \leq 0 \leq \Gamma_1 \leq \Gamma_3$. Then we can rewrite the previous equation as

$$(4.14) \quad \dot{\gamma}^2 = 4\mu_1 \mu_2 \mu_3 (\gamma - \Gamma_1)(\gamma - \Gamma_2)(\gamma - \Gamma_3).$$

Proposition 4.2. $\gamma(t) = \Gamma_2 - (\Gamma_2 - \Gamma_1) \operatorname{sn}^2(rt, k)$ is a solution of (4.14) provided

$$r^2 = \mu_1 \mu_2 \mu_3 (\Gamma_3 - \Gamma_2), \quad k^2 = \frac{\Gamma_2 - \Gamma_1}{\Gamma_2 - \Gamma_3}$$

where sn is the Jacobi elliptic sn-noidal function.

Proof. The proof is a straightforward computation using the basic properties of the Jacobi elliptic functions (for details see [7]). \square

From this proposition and (4.7) we derive expressions for R_j^2

$$(4.15) \quad R_j^2 = \mu_j (\gamma - \gamma_j) = \mu_j ((\Gamma_2 - \gamma_j) - (\Gamma_2 - \Gamma_1) \operatorname{sn}^2(rt, k))$$

where $\gamma_j = -1/\mu_j$.

As promised in the proof of Theorem D we now provide explicit solutions for the $J = 0$ case.

Proposition 4.3. For each $\theta \in [0, 2\pi)$, there exists a family of θ -special Legendrian immersions $u_{\alpha,0} : \mathbb{R}^2 \rightarrow S^5(1)$, for $\alpha \in [0, 1]$, whose Gauss curvature K satisfies (4.20) and (4.21) (and hence are all distinct). Moreover, $u_{0,0}$ gives rise to a θ -special Legendrian sphere and is the only member of the family $u_{\alpha,J}$ to do so.

Proof. In the case $J = 0$ we know explicitly the values of the Γ_i

$$\Gamma_i = \gamma_i = -\frac{1}{3\mu_i}, \quad i = 1, 2, 3$$

and hence

$$(4.16) \quad r^2 = (1 + 2\alpha), \quad k^2 = \frac{1 - \alpha^2}{1 + 2\alpha}.$$

Equation (4.15) specializes to

$$(4.17) \quad R_1 = \mu_1(\gamma_2 - \gamma_1) \operatorname{cn}(rt, k)$$

$$(4.18) \quad R_2 = \mu_2(\gamma_1 - \gamma_2) \operatorname{sn}(rt, k)$$

$$(4.19) \quad R_3 = \mu_3(\gamma_2 - \gamma_3) \operatorname{dn}(rt, k).$$

Define $u_{\alpha,0}$ by the formula

$$u_{\alpha,0}(s, t) = e^{As}(e^{i(\theta+\pi/2)}R_1(t), R_2(t), R_3(t))$$

where as previously we set $A = i \operatorname{diag}(1, \alpha, -1 - \alpha)$ for $\alpha \in [0, 1]$. Then u is a θ -special Legendrian immersion invariant under e^{As} .

To find the extreme values taken on by the Gauss curvature, note that in the case $J = 0$ we have $y_{\min} = -\lambda_2\lambda_3 = \alpha(1+\alpha)$ and $y_{\max} = -\lambda_1\lambda_3 = 1+\alpha$. Thus

$$(4.20) \quad K_{\min} = 1 + \frac{2\lambda_1^2}{\lambda_2\lambda_3} = 1 - \frac{2}{\alpha(1+\alpha)}$$

and

$$(4.21) \quad K_{\max} = 1 + \frac{2\lambda_2^2}{\lambda_1\lambda_3} = 1 - \frac{2\alpha^2}{1+\alpha}.$$

From (4.16) we see that $k^2 \rightarrow 1$ as $\alpha \rightarrow 0$, and $k^2 \rightarrow 0$ as $\alpha \rightarrow 1$. In these two limits sn reduces to \tanh and \sin respectively. Thus in the limiting case $\alpha = 0, J = 0$ we have

$$\gamma = -\frac{1}{6} + \frac{1}{2}\tanh^2 t, \quad R_1 = R_3 = \frac{1}{\sqrt{2}}\operatorname{sech} t, \quad R_2 = \tanh t.$$

Finally, one can show that in order for any immersion of the form $u(s, t) = e^{As}z(t)$ to describe a harmonic sphere, the limit of $z(t)$ as $t \rightarrow \pm\infty$ must be a fixed point of the action e^{As} [13]. Moreover, all the conserved quantities of the Neumann system must also be zero (since they are zero at a fixed point). For $A \in \mathfrak{su}(3)$ as above, e^{As} has nonzero fixed points if and only if $\alpha = 0$, in which case any point of the form $(0, z_2, 0) \in \mathbb{C}^3$ is fixed. From equation (4.6), all three angular momenta J_j are zero if and only if $J = 0$. Thus $u_{0,0}$ is the only $u_{\alpha,J}$ which could describe a minimal sphere. In this case u (in the 0-special Legendrian case) has the explicit form

$$(4.22) \quad u(s, t) = \left(\frac{1}{\sqrt{2}}ie^{is}\operatorname{sech} t, \tanh t, \frac{1}{\sqrt{2}}e^{-is}\operatorname{sech} t \right)$$

and we can see directly that the 2-sphere described is the intersection of the plane

$$-i\bar{z}_1 = z_3, \quad \operatorname{Im} z_2 = 0$$

with the 5-sphere (and hence is totally geodesic). \square

5. PERIODICITY CONDITIONS

In order to analyze the periodicity of the immersions $u_{\alpha,J}$ we need the following lemma whose proof is a short computation (see [7] for full details).

Lemma 5.1. *For $J \neq 0$ the sum of the angles $\sum \theta_i$ and $\dot{\gamma}$ satisfy*

$$(5.1) \quad \dot{\gamma}(t) = 2J \tan \left(\sum \theta_i(t) - \theta_i(0) \right).$$

Since we chose γ so that $\dot{\gamma}(0) = 0$, this lemma has the following obvious corollary:

Corollary 5.2. *If T is the period of γ then $\sum \theta_i(T) = \sum \theta_i(0) + n\pi$, for some integer n .*

The previous lemma is also useful in verifying what conditions on $\theta_1(0)$ ensure that the immersions $u_{\alpha,J}$ are θ -special Legendrian. For this we need to compute β_θ restricted to the cone on u . At a point (x, s, t) on the cone

$$\beta_\theta = x^2 \text{Im} \left(e^{i\theta} \det_{\mathbb{C}}(u, u_s, u_t) \right) = x^2 \text{Im}(e^{i\theta} \det_{\mathbb{C}}(z, Az, \dot{z})).$$

If $J = 0$, so that θ_j are all constant we have

$$\det_{\mathbb{C}}(z, Az, \dot{z}) = ie^{i \sum \theta_j(0)} |Az|^2$$

and hence the immersion is θ -special Legendrian where θ depends only on the initial sum of the angles θ_j . For example, if we choose $\theta_1(0) = \pi/2$ or $\theta_1(0) = 3\pi/2$ (and $\theta_2(0) = \theta_3(0) = 0$) then the cones are 0-SLG.

If $J \neq 0$ a short computation using Lemma 5.1 shows that

$$\det_{\mathbb{C}}(z, Az, \dot{z}) = \frac{i|Az|^2 e^{i \sum \theta_i(0)}}{R_1^2 R_2^2 R_3^2} (J + i\dot{\gamma}/2)(\dot{\gamma}/2 + iJ) = -|Az|^2 e^{i \sum \theta_i(0)}$$

so that now choosing $\theta_1(0) = 0$ or $\theta_1(0) = \pi$ gives 0-special Legendrian immersions.

Suppose that (σ, τ) is a period of $u(s, t) = e^{As} z(t)$, i.e.

$$(5.2) \quad u(s + \sigma, t + \tau) = u(s, t) \quad \forall s, t.$$

Then the periodicity properties of u are characterized by

Proposition 5.3.

(a) (σ, τ) is a period of $u_{\alpha,J}$ implies τ is an integer multiple of $T_{\alpha,J}$, the basic period of $y_{\alpha,J} = |Az|^2$

(b) If u admits two independent periods then it admits a period of the form $(\sigma, 0)$

(c) u admits a period of the form $(\sigma, 0)$ if and only if $\alpha \in \mathbb{Q}$

(d) u admits two independent periods if and only if

$$(5.3) \quad \alpha, \quad \frac{1}{2\pi}(\alpha\theta_1(T) - \theta_2(T)) \in \mathbb{Q}.$$

Proof. (a) Differentiating (5.2) with respect to s and taking the norm of both sides implies $|Az(t + \tau)| = |Az(t)|$.

(b) If the periods are (σ_1, n_1T) and (σ_2, n_2T) then $(n_1\sigma_2 - n_2\sigma_1, 0)$ is also a period.

(c) $(\sigma, 0)$ is a period implies $e^{i\sigma\lambda_j} = 1$, for $j = 1, 2, 3$. So $\sigma\lambda_j \in 2\pi\mathbb{Z}$. In particular, $\alpha = \frac{\lambda_2}{\lambda_1} \in \mathbb{Q}$. Conversely if $\alpha = \frac{m}{n}$ then $(\frac{2n\pi}{\lambda_1}, 0)$ is a period.

(d) If u admits two independent periods then α is rational by (b) and (c). By (a) any period is of the form (σ, mT) . Now since $\dot{\theta}_j$ is T -periodic we have $\theta_j(nT) = n\theta_j(T)$. Then periodicity with respect to (σ, mT) is equivalent to $e^{i\lambda_j\sigma + im\theta_j(T)} = 1$. Hence $\sigma\lambda_j + m\theta_j(T) \in 2\pi\mathbb{Z}$, for $j = 1, 2, 3$. Together with rationality of α this implies $\alpha\theta_1(T) - \theta_2(T) \in 2\pi\mathbb{Q}$.

Conversely, by (c) the rationality of α gives us one period $(\sigma_1, 0)$. From above (σ, mT) is a period if and only if $\sigma\lambda_j + m\theta_j(T) \in 2\pi\mathbb{Z}$, $j = 1, 2, 3$. By assumption $\frac{1}{2\pi}(\alpha\theta_1(T) - \theta_2(T)) = \frac{M}{N}$ for some integers M and N . With $\sigma = -\frac{2N\theta_1(T)}{\lambda_1}$ and $m = 2N$ the period condition becomes

$$2N \left(-\frac{\lambda_j}{\lambda_1} \theta_1(T) + \theta_j(T) \right) \in 2\pi\mathbb{Z}, \quad j = 1, 2, 3.$$

For $j = 1$ this condition is trivial, while it holds for $j = 2$ because

$$2N \left(-\frac{\lambda_2}{\lambda_1} \theta_1(T) + \theta_2(T) \right) = -4N\pi (\alpha\theta_1(T) - \theta_2(T)) = -4M\pi$$

Since $\sum \lambda_i = 0$ and by Corollary 5.2, $\theta_3(T) = n\pi - \theta_1(T) - \theta_2(T)$, we have

$$2N \left(-\frac{\lambda_3}{\lambda_1} \theta_1(T) + \theta_3(T) \right) = 2N (\alpha\theta_1(T) - \theta_2(T) + n\pi) = 4M\pi + 2\pi Nn.$$

So the $j = 3$ period condition also holds, and hence $(-\frac{2N\theta_1(T)}{\lambda_1}, 2NT)$ is a second period of u . \square

Two cases of the previous proposition are particularly interesting: when $J = 0$ or $\alpha = 0$. For the case $J = 0$ we prove the following result which implies Theorem C and part (i) of Theorem E.

Proposition 5.4. *For $\alpha \in \mathbb{Q} \cap (0, 1]$, the immersion $u_{0,\alpha}$ is doubly periodic and hence gives rise to a minimal Legendrian torus. Further, let $\alpha = \frac{m}{n}$, where $m < n \in \mathbb{N}$ and $(m, n) = 1$. If mn is even, then the period lattice of $u_{\alpha,0}$ is rectangular with basis $\omega_1 = (2n\pi, 0)$, $\omega_2 = (0, 4\text{Ke}(k)/r)$. Otherwise the period lattice is not rectangular and is generated by $\omega_1 = (2n\pi, 0)$ and $\omega_3 = (n\pi, 2\text{Ke}(k)/r)$. In either case each such torus $T_{m,n}$ is embedded and its Gauss curvature satisfies (4.20) and (4.21).*

Notation in the proposition: k and r are defined as a functions of α by (4.16), and Ke is the complete elliptic integral defined by

$$\text{Ke}(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

(the period of $\text{sn}(t, k)$ is $4\text{Ke}(k)$).

Proof. Since $J = 0$, the θ_i are constant and the second condition of part (d) of the previous proposition is superfluous. Thus the immersion is doubly periodic if and only if $\alpha \in \mathbb{Q}$. Let $\alpha = \frac{m}{n}$. It is easy to see that $\omega_1 = (2n\pi, 0)$ and $\omega_2 = (0, 4\text{Ke}/r)$ belong to the period lattice of $u_{\alpha,0}$. To find the full period lattice it is sufficient to find all periods in the rectangle R formed by $0, \omega_1, \omega_2$ and $\omega_1 + \omega_2$. Let (σ, τ) be such a period. By part (a) of the previous proposition τ must be a integer multiple of $2\text{Ke}/r$, the basic period of y . It is easy to see that the smallest period of the form $(\sigma, 0)$ occurs when $\sigma = 2n\pi$ and so we need only deal with the case $\tau = 2\text{Ke}/r$. Using the fact that $\text{cn}(t + 2\text{Ke}) = -\text{cn}(t)$, $\text{sn}(t + 2\text{Ke}) = -\text{sn}(t)$, $\text{dn}(t + 2\text{Ke}) = \text{dn}(t)$ we find that $(\sigma, 2\text{Ke}/r)$ is a period if and only iff σ satisfies

$$(5.4) \quad e^{i\sigma} = -1, \quad e^{i\sigma\alpha} = e^{i\sigma m/n} = -1, \quad e^{-i(1+\alpha)} = 1.$$

Clearly the third equation is implied by the first two. Moreover, the first equation implies $e^{im\sigma} = (-1)^m$, whereas the second implies $e^{im\sigma} = (-1)^n$. Hence if either m or n is even (both cannot be even since we assumed $(m, n) = 1$) then these two equations are inconsistent. Thus there are no further periods and the period lattice is generated by ω_1 and ω_2 . If both m and n are odd, then one can check that $\sigma = n\pi$ is the unique solution in $[0, 2n\pi)$. Hence $w_3 = (n\pi, 2\text{Ke}/r)$ is the only new period in the rectangle R and in this case the period lattice is generated by ω_1 (or ω_2) and ω_3 .

Let us show embeddedness in the case where one of m, n is even. The other case is similar, but a little more involved since the period lattice is not rectangular. We need to show that if $s, \tilde{s} \in [0, 2n\pi)$ and $t, \tilde{t} \in [0, 4\text{Ke}/r)$ and $u(s, t) = u(\tilde{s}, \tilde{t})$ then $s = \tilde{s}$ and $t = \tilde{t}$. From our explicit formulae for R_i we see that $u(s, t) = u(\tilde{s}, \tilde{t})$ is equivalent to

$$(5.5) \quad e^{is} \text{cn}(t/r) = e^{i\tilde{s}} \text{cn}(\tilde{t}/r)$$

$$(5.6) \quad e^{is} \text{sn}(t/r) = e^{i\tilde{s}} \text{sn}(\tilde{t}/r)$$

$$(5.7) \quad e^{-i(1+\alpha)s} \text{dn}(t/r) = e^{i(1+\alpha)\tilde{s}} \text{dn}(\tilde{t}/r).$$

Certainly this implies $|\text{cn } t/r| = |\text{cn } \tilde{t}/r|$, which implies there exists some $T \in [0, \text{Ke}]$ such that $rt, r\tilde{t} \in \{T, 2\text{Ke} - T, 2\text{Ke} + T, 4\text{Ke} - T\}$. If t and \tilde{t} are distinct, there are essentially two different cases, depending on whether $\text{cn } t = -\text{cn } \tilde{t}, \text{sn } t = -\text{sn } \tilde{t}$ or $\text{cn } t = \pm \text{cn } \tilde{t}, \text{sn } t = \mp \text{sn } \tilde{t}$. In the first case the three equations above reduce to

$$e^{i\sigma} = -1, \quad e^{i\alpha\sigma} = -1, \quad e^{-i\sigma(1+\alpha)} = 1$$

where $\sigma = s - \tilde{s}$. That is, we have the same equations as occurred in the periodicity part of the proof. Since we assumed one of m and n was even, the first two equations are inconsistent unless $t = \tilde{t}$ in which case $s = \tilde{s}$ is also forced. In the second case the equations reduce to

$$e^{i\sigma} = \pm 1, \quad e^{i\alpha\sigma} = \mp 1, \quad e^{-i\sigma(1+\alpha)} = 1.$$

Clearly, the first two equations are inconsistent with the third one. □

In the case $J \neq 0$, $\alpha = 0$, the conditions in part (d) reduce to $\theta_2(T) \in \mathbb{Q}$. Using the explicit expressions given in the previous chapter and properties of elliptic functions one can show that viewed as a function of J , $\theta_2(T)$ is strictly monotone. Part (ii) of Theorem E follows.

We conclude with the following result which demonstrates the sharpness of the pinching results on minimal Legendrian immersions given in parts (ii) and (iii) of Theorem 2.7.

Theorem 5.5. *For any $\epsilon > 0$ there exists an embedded minimal Legendrian torus T in S^5 which is not flat, but for which $\sup_{x \in T} |K(x)| < \epsilon$, where K is the Gauss curvature of T .*

Proof. Consider an immersion $u_{\alpha,J}$ with $J = 0$ and $\alpha = 1 - \delta$. From (4.20) and (4.21) the minimum and maximum values of the Gauss curvature are given by

$$(5.8) \quad K_{\min}(u_{1-\delta,0}) = -\frac{\delta(3+\delta)}{(1-\delta)(2-\delta)}$$

and

$$(5.9) \quad K_{\max}(u_{1-\delta,0}) = \frac{\delta(3-2\delta)}{2-\delta}.$$

Certainly for $\delta < \frac{1}{2}$ we have $|K_{\min}| < 7\delta$ and similarly for K_{\max} . Since $u_{1-\delta,0}$ gives rise to an embedded minimal Legendrian torus whenever $\alpha \in \mathbb{Q}$, just choose $\delta \in \mathbb{Q} \cap (0, \epsilon/7)$ and the result is proved. \square

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